§1 HAMILTONIAN MECHANICS

Version 1.6

Notiztitel

We begin with a special case of a system of CLASSICAL MECHANICS where the configuration space ("Ortsraum") is an open subset $\Omega \subset \mathbb{R}^n$ of $\mathbb{R}^n$ ($n \in \mathbb{N}$, $n \geq 0$):

- $\Omega \subset \mathbb{R}^n$ open,
- $P := \Omega \times \mathbb{R}^n \cong T^*\Omega$ PHASE SPACE,
- $H \in \mathcal{C}^\infty(P)$ HAMILTONIAN FUNCTION.

The EQUATIONS OF MOTION ("Bewegungsgleichungen") are

$$\dot{q} = \frac{\partial H}{\partial p}, \text{ i.e. } \dot{q}^j = \frac{\partial H}{\partial p_j}, \; j = 1, 2, \ldots n$$

$$\dot{p} = -\frac{\partial H}{\partial q}, \text{ i.e. } \dot{p}_j = \frac{\partial H}{\partial q^j}, \; j = 1, 2, \ldots n$$

Here, $q = (q^1, \ldots q^n)$ are the coordinates in Ω ("Ortskoordinaten") and $p = (p_1, \ldots p_n)$ are the coordinates of the cotangent space $T_q^*\Omega \cong \mathbb{R}^n$ ("Impulskoordinaten").

The equations of motion are called CANONICAL EQUATIONS or HAMILTONIAN EQUATIONS, and (P, H) is called a (FLAT) HAMILTONIAN SYSTEM (with $n = \dim \Omega$ degrees of freedom).

An example of such a flat Hamiltonian system is the n-dimensional harmonic oscillator, where

$$\Omega = \mathbb{R}^n \quad \text{and}$$
$$H(q,p) := \frac{1}{2}\left(\|p\|^2 + \|q\|^2\right) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + (q^j)^2$$

with canonical equations

$$\dot{q} = p \quad \text{and} \quad \dot{p} = -q \, .$$

The canonical equations can be written in the form

$$(\dot{q},\dot{p}) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$$

In this form they look like a dynamical system of the type

$$\dot{x} = A(x) \, ,$$

where the vector field A is given as the gradient $A = \nabla f = \frac{\partial f}{\partial x}$ of a $\mathcal{C}^\infty$-function $f : X \to \mathbb{R}$, $X \subset \mathbb{R}^m$ open.

If we define the SYMPLECTIC STRUCTURE on the phase space $P = T^*\Omega \cong \Omega \times \mathbb{R}^n$ by the map

$$\sigma : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n} \, , \quad (q,p) \longmapsto (p,-q) \, ,$$

given by the block matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

acting as

$$\begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix},$$

and if we define the SYMPLECTIC GRADIENT $\nabla^\sigma F$ of $F \in \mathcal{C}^\infty(P)$ to be

$$\nabla^\sigma F = \sigma \circ \nabla F = \left(\frac{\partial F}{\partial p}, -\frac{\partial F}{\partial q}\right),$$

then our canonical equations obtain the form

$$\dot{a} = \nabla^\sigma H(a) \quad \text{for } a = (q,p) \in P = \Omega \times \mathbb{R}^n$$

or

$$\boxed{\dot{a} = X_H(a)},$$

if we denote the vector field $\nabla^\sigma H$ by X_H.

X_H is called the HAMILTONIAN VECTOR FIELD associated with H.

Because of $\sigma^2 = \sigma \circ \sigma = -\mathrm{id}_{\mathbb{R}^{2n}}$ the map σ is also called the SYMPLECTIC INVOLUTION.

The symplectic structure on $P = T^*\Omega$ and, in particular on $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, can also be given by the symplectic form or by the Poisson bracket [*] as is explained in the following:

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* Übung: Prove equivalent descriptions of the symplectic structure by $\sigma, \omega, \{\,,\,\}$ and generalize to manifolds (see below).

The SYMPLECTIC FORM ω on the tangent space

$$T_a P \cong \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n} \quad \text{at } a \in P$$

is

$$\omega = dq^j \wedge dp_j = \sum_{j=1}^{n} dq^j \wedge dp_j \quad \text{(Einstein summation).}$$

Hence, the bilinear and alternating map ω

$$\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}$$

is given by

$$\omega(X, \bar{X}) = X^j \bar{Y}_j - \bar{X}^j Y_j ,$$

where $X = (X^1, \dots X^n, Y_1, \dots, Y_n)$, $\bar{X} = (\bar{X}^1, \dots \bar{X}^n, \bar{Y}_1, \dots, \bar{Y}_n) \in \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ in the standard unit coordinates of $\mathbb{R}^n \oplus \mathbb{R}^n$.
ω can also be described by

$$\omega(X, \bar{X}) = X^T \sigma \bar{X}$$

("T" denotes transposition).

Finally, the POISSON BRACKET $\{F, G\}$ of two functions ("observables") $F, G \in \mathcal{C}^\infty(P)$ is defined by

$$\{F, G\} := \omega(X_F, X_G),$$

which leads to the well-known expression

$$\{F, G\} = \frac{\partial F}{\partial q^j} \cdot \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q^j} = \frac{\partial F}{\partial q} \cdot \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \cdot \frac{\partial G}{\partial q}.$$

REMARK: Other sign conventions are used, for example, for the symplectic form:

$$dp_j \wedge dq^j$$

which is $-\omega$ in our notation.

A straightforward but remarkable property is the following (EQUATION OF MOTION IN POISSON FORM):

(1.1) <u>PROPOSITION:</u> A curve $a : [t_0, t_1] \longrightarrow P$ is a solution of $\dot{a} = X_H(a)$ if and only if

$$\dot{F} = \{F, H\} \quad \text{for all } F \in \mathcal{C}^\infty(P),$$

i.e.

$$\frac{d}{dt} F(a(t)) = \{F(a(t)), H(a(t))\}, \quad t \in [t_0, t_1].$$

□ <u>Proof.</u> If $a(t) = (q(t), p(t))$ is solution of $\dot{a} = X_H(a)$ we have

$$\dot{F} = \frac{d}{dt} F(a(t)) = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q}.$$

Hence, $\dot{F} = \{F, H\}$.

The converse follows by choosing p_j for F and q^j for F : $\dot{p} = -\frac{\partial H}{\partial q}$ & $\dot{q} = \frac{\partial H}{\partial p}$. □

What we have described so far is in some sense only the local version of classical mechanics. For the global case which is needed for the program of geometric quantisation we have to extend the above concepts to general manifolds. Note, that in classical mechanics the reduction of degrees of freedom by first integrals ("Bewegungskonstante") or by symmetry consideration naturally leads to general manifolds.[*]

MANIFOLDS:

A manifold (in this course) is always a smooth (i.e. $\mathcal{C}^\infty$) manifold with countable[1] topology. In many cases the manifold is also assumed to be connected. The charts of the manifold M are written in the following way:

$$q : \mathcal{U} \to V,$$

where $\mathcal{U} \subset M$ is an open subset of M and $V \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, $q = (q^1, \dots, q^n)$. The smooth functions $q^j : \mathcal{U} \to \mathbb{R}$ are the coordinates (given by the chart q), and $n \in \mathbb{N}$, $n \geq 0$, is the dimension of M. (The complete definition of the notion of a manifold of dimension n requires a Hausdorff space M together with an atlas of $\mathcal{C}^\infty$-compatible charts with models in $\mathbb{R}^n$.)

In principle, we are also interested in ∞-dimensional manifolds with models in a fixed Hilbert space

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* Übung: Describe reduction in the case of one constant of motion. Give explicit examples, i.e. pendulum, or rigid body.

[1] parakompact is enough!

E, in a Banach space E or even a Fréchet space E. In this course, however, to simplify matters, we concentrate on the finite dimensional case.

The relevant manifolds occurring in geometry & physics are:
- open subsets $\Omega \subset \mathbb{R}^n$,
- tangent and cotangent bundles $T\Omega, T^*\Omega$,
- $SO(3)$ and other Lie groups,
- submanifolds of the above,
- quotients of the above. [*]

<u>Notations</u>: For manifolds M, N:

$$\mathcal{E}(M,N) := \{ f : M \to N : f \text{ smooth} \},$$
$$\mathcal{E}(M) = \mathcal{E}(M,\mathbb{R}) \quad (\text{or later } \mathcal{E}(M) = \mathcal{E}(M,\mathbb{C})).$$

Each curve $x \in \mathcal{E}(I, M)$, $I \subset \mathbb{R}$ open interval with $0 \in I$, $x(0) = a \in M$, determines a tangent vector $[x]_a \in T_a M$, which is the equivalence class of (germs of) curves in M through $a \in M$ given by the equivalence relation

$$x \sim_a y \ \left(\text{i.e. } [x]_a = [y]_a\right) \iff \frac{d}{dt}(\varphi \circ x)\Big|_{t=0} = \frac{d}{dt}(\varphi \circ y)\Big|_{t=0},$$

where $y \in \mathcal{E}(I, M)$ with $y(0) = a$.

The TANGENT SPACE $T_a M$ at $a \in M$ is the space

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*Übung: Describe explicitly (e.g. by the charts) the projective spaces $\mathbb{P}^n(\mathbb{C})$, $\mathbb{P}^n(\mathbb{R})$ as quotient manifolds of $\mathbb{C}^{n+1} \setminus \{0\}$, $\mathbb{R}^{n+1} \setminus \{0\}$.

of all equivalence classes $[x]_a$ with the obvious structure of a vector space over $\mathbb{R}$.

The TANGENT BUNDLE ("Tangentialbündel")

$$TM = \bigcup_{a \in M} T_a M$$

together with the projection

$$\tau : TM \to M \quad , \quad \xi \in T_a M \mapsto a,$$

has a natural structure of a $2n$-dimensional manifold where τ is smooth and the fibres $\tau^{-1}(a) = T_a M$ are vector spaces $\cong \mathbb{R}^{2n}$: TM is a vector bundle.

A map $f : M \to N$ between manifolds M, N is SMOOTH if for each $a \in M$ there are (smooth) charts $\varphi : U \to V \subset \mathbb{R}^m$ with $a \in U$ and $\psi : U' \to V' \subset \mathbb{R}^n$ with $f(a) \in U'$ such that

$$\psi \circ f \circ \varphi^{-1} : V \to V'$$

is smooth (i.e. $\in \mathcal{C}^\infty(V, V')$, $V \subset \mathbb{R}^m$, $V' \subset \mathbb{R}^n$).

For a smooth map $f : M \to N$ between manifolds M, N the TANGENT MAP or DERIVATIVE at a point $a \in M$ is given by

$$T_a f : T_a M \to T_b N \quad , \quad T_a f([x]_a) = [f \circ x]_b ,$$

where $b = f(a) \in N$. The $T_a f$, $a \in M$, define the TANGENT MAP (DERIVATIVE or JACOBIAN)

$$Tf : TM \to TN, \qquad v \longmapsto T_a f(v) \ , \ v = \tau(a),$$

which turns out to be a smooth map, compatible with the projections (i.e. $f \circ \tau_M = \tau_N \circ Tf$) and $\mathbb{R}$-linear on the fibres. (Hence, Tf is a vector bundle homomorphism over f.) The compatibility can also be expressed by stating that the diagram

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\tau_M"'] & TN \arrow[d, "\tau_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

is commutative: $f \circ \tau_M = \tau_N \circ Tf$.

Similarly, we have the COTANGENT BUNDLE ("Kotangentialbündel")

$$T^*M = \bigcup_{a \in M} T_a^*M,$$
$$T_a^*M := \mathrm{Hom}(T_aM, \mathbb{R}) = (T_aM)^*$$

with projection $T^*M \to M$, denoted by τ or τ^*. T^*M is the vector bundle of 1-forms.

In addition to the two vector bundles TM and T^*M we also need the bundles of p-forms for $p = 2, 3, \dots$.

$$\Lambda_a^p M := \{\alpha : T_aM^p \to \mathbb{R} \mid \alpha \text{ p-linear over } \mathbb{R} \text{ \& alternating}\}$$
$$\Lambda^p M = \bigcup_{a \in M} \Lambda_a^p M \qquad (\Lambda^1 M = T^*M)$$

with smooth projections
$$\tau : \Lambda^p M \longrightarrow M. \text{[*]}$$

The VECTOR FIELDS on M are the smooth sections
$$X : M \longrightarrow TM \;, \quad X(a) \in T_a M \;, \quad \text{i.e.}$$
$$X \in \mathcal{E}(M, TM) \text{ with } \quad \tau \circ X = id_M .$$

We denote the set of vector fields on M by $\mathfrak{V}(M)$. $\mathfrak{V}(M)$ is a vector space over $\mathbb{R}$ and a module over the ring $\mathcal{E}(M)$ by
$$(X+Y)(a) := X(a) + Y(a) \;, \quad X, Y \in \mathfrak{V}(M), \text{ and}$$
$$f X(a) := f(a)\, X(a) \quad , \quad f \in \mathcal{E}(M), X \in \mathfrak{V}(M).$$

It turns out (see remark after proposition 1.3) that $\mathfrak{V}(M)$ is, in addition, a Lie algebra over $\mathbb{R}$, – this explains the notation $\mathfrak{V}(M)$ with the "german" V.

Similarly, the DIFFERENTIAL FORMS OF DEGREE p (or p-FORMS) on M are defined by
$$\Omega^p(M) := \{ \alpha \in \mathcal{E}(M, \Lambda^p M) \mid \tau \circ \alpha = id_M \}$$
for $p = 1, 2, \ldots$. $\Omega^p(M)$ is an $\mathcal{E}(M)$-module as well.

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*Übung: Describe $\mathcal{C}^\infty$-structure of TM, $\Lambda^p M$ by bundle charts, see below.

(1.2) <u>PROPOSITION:</u> $\Omega^p(M)$ are $\mathcal{E}(M)$-modules

$$\Omega^1(M) \cong \mathcal{V}(M)^* \quad \text{as } \mathcal{E}(M)\text{-modules},$$
$$\Omega^p(M) \cong \Lambda^p \mathcal{V}(M) \quad \text{" " " }. \; [*]$$

Here, for a module V over a ring R we set

$$\Lambda^p_R V := \{ \beta : V^p \to R \mid \beta \text{ p-linear over } R \text{ and alternating} \}$$

$$\Lambda^p V := \Lambda^p_R V =: \mathrm{Alt}^p(V,R) \,.$$

<u>Differential and Lie derivative:</u>

Given $f \in \mathcal{E}(M)$ and $X = [x]_a \in T_aM$ we define

$$df(a)(X) := \frac{d}{dt}(f \circ x)\Big|_{t=0} \in \mathbb{R}$$

to obtain $df(a) \in (T_aM)^* = T_a^*M$ and $df \in \Omega^1(M)$. The $\mathbb{R}$-linear map

$$d : \mathcal{E}(M) \longrightarrow \Omega^1(M)$$

is the DIFFERENTIAL of $f \in \mathcal{E}(M)$. Note, that

$$d(fg) = f\,dg + g\,df \quad \text{for } f,g \in \mathcal{E}(M),$$

i.e. d is a DERIVATION.

For a vector field $X \in \mathcal{V}(M)$ the LIE DERIVATIVE ("Richtungsableitung") in the direction X is the map

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* Übung: Prove!

$$L_X : \mathcal{E}(M) \longrightarrow \mathcal{E}(M)$$

defined by

$$L_X f := df(X) \quad , \quad f \in \mathcal{E}(M),$$

and denoted also by $Xf := L_X f$.

For a point $a \in M$ the value $L_X f(a) \in \mathbb{R}$ of the function $L_X f$ is

$$L_X f(a) = \frac{d}{dt} f(x(t)) \Big|_{t=0},$$

where the curve x in M with $x(0) = a$ represents the tangent vector $X(a)$, i.e.

$$X(a) = [x]_a \in T_a M .$$

L_X is $\mathbb{R}$-linear and it is also a derivation:

$$L_X(fg) = f L_X g + g L_X f \ , \ \text{for } g, f \in \mathcal{E}(M).$$

In local coordinates: Let $\varphi = (\varphi^1, \ldots \varphi^n) : U \to V$, be a chart, i.e. $U \subset M$ open, $V \subset \mathbb{R}^n$ open and φ a diffeomorphism. With respect to the standard unit vectors $e_j = (\delta_{ij})_{i=1,\ldots n} \in \mathbb{R}^n$ of $\mathbb{R}^n$ we obtain by

$$E_j(a) := [\varphi^{-1}(\varphi(a) + t e_j)]_a \in T_a M$$

a basis of $T_a M$ for each $a \in U$.
Let $v \in \tau^{-1}(U) = TU \subset TM$ with $[x]_a = v$, where $a = \tau(v)$.
Then the bundle chart $\tilde{\varphi} : \tau^{-1}(U) \longrightarrow V \times \mathbb{R}^n$ associated with φ is defined by

$$\tilde{\varphi}(v) = \left(\varphi(a), \frac{d}{dt}(\varphi \circ x)\big|_{t=0}\right),$$

and we have

$$\tilde{\varphi}(E_j(a)) = (\varphi(a), e_j) \quad , \; j=1, \dots n .$$

$E_j : \mathcal{U} \to T\mathcal{U}$ is a vector field on $\mathcal{U}$. Its action is

$$L_{E_j} f(a) = df(E_j)(a) = \frac{d}{dt}\left(f \circ \bar{\varphi}^1(\varphi(a) + t e_j)\big|_{t=0}\right)$$
$$= \frac{\partial}{\partial q_j}(f \circ \bar{\varphi}^1)(\varphi a).$$

Therefore, the vector field E_j is denoted by

$$\frac{\partial}{\partial q_j} := E_j ,$$

and we write

$$L_{E_j} f(a) = \frac{\partial f}{\partial q_j}(a) .$$

For a general vector field $X \in \mathcal{W}(\mathcal{U})$ we get the unique representation

$$X = X^j E_j = X^j \frac{\partial}{\partial q_j}$$

with coefficients $X^j \in \mathcal{E}(\mathcal{U})$.

The Lie derivative $L_X f$ for $X \in \mathcal{W}(\mathcal{U})$ & $f \in \mathcal{E}(\mathcal{U})$ yields the form

$$L_X f = X^j \frac{\partial f}{\partial q_j} ,$$

and the differential df is

$$df = \frac{\partial f}{\partial q_j} dq^j .$$

In particular, $dq^k\left(\frac{\partial}{\partial q^j}\right) = \delta_j^k$ (Kronecker).

<u>Local expressions:</u>

Let q^j, $j=1,2,\dots n$, be local coordinates on a manifold M given by a chart $q: U \to V \subset \mathbb{R}^n$, $U \subset M$ open. We have stated above that a vector field $X: U \to TM|_U = TU$ has a unique representation

$$X = X^j \frac{\partial}{\partial q^j} \quad, \text{with } X^j \in \mathcal{E}(U),$$

with respect to the vector fields

$$\frac{\partial}{\partial q^j} : U \to TU \quad, \; j = 1, \dots n.$$

In fact, $X^j := dq^j(X)$.

Thus, $\mathcal{V}(U)$ is an $\mathcal{E}(U)$-module with basis $\left\{\frac{\partial}{\partial q^j} : 1 \le j \le n\right\}$.

Similarly, $\Omega^1(U)$ is a free $\mathcal{E}(U)$-module of rank n with $\{dq^j : 1 \le j \le n\}$ as a basis: Every $\alpha \in \Omega^1(U)$ has a unique representation

$$\alpha = \alpha_j dq^j \quad \text{with } \alpha_j \in \mathcal{E}(U),$$

given by $\alpha_j := \alpha\left(\frac{\partial}{\partial q^j}\right)$.

The corresponding result is true for general forms (and tensors): Any $\omega \in \Omega^k(U)$ can be written uniquely as

$$\omega = \omega_{i_1 \dots i_k} dq^{i_1} \wedge \dots \wedge dq^{i_k} ,$$

with

$$\omega_{i_1 \dots i_k} = \omega\left(\frac{\partial}{\partial q^{i_1}}, \dots , \frac{\partial}{\partial q^{i_k}}\right) \in \mathcal{E}(U), \; i_j \in \{1,2,\dots n\}.$$

For a commutative (and associative) algebra A over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we denote

$$\mathrm{Der}\, A := \{ D : A \to A \mid D \ \mathbb{K}\text{-linear} \ \& \ D(fg) = D(f)g + fD(g) \text{ for all } f, g \in A \}.$$

Der A, the set of DERIVATIONS, is a vector space over $\mathbb{K}$ with respect to pointwise addition and multiplication by scalars $\lambda \in \mathbb{K}$. Moreover, it is easy to see that the commutator $[D, D'] = DD' - D'D$ of two derivations $D, D' \in \mathrm{Der}\, A$ is again a derivation: $[D, D'] \in \mathrm{Der}\, A$. Hence, Der A with $[\ ,\ ]$ is a Lie algebra over $\mathbb{K}$.

(1.3) PROPOSITION: $L : \mathcal{V}(M) \longrightarrow \mathrm{Der}(\mathcal{E}(M))$ is bijective (and $\mathbb{R}$-linear). [*]

Remark: As a consequence, the $\mathbb{R}$-vector space $\mathcal{V}(M)$ with the commutator $[X, Y] := Z$, where Z is given by $L_Z = L_X L_Y - L_Y L_X$, is a Lie algebra over $\mathbb{R}$.

Exterior derivative:

Recall, that there is a natural isomorphism

$$\Omega^p(M) \cong \mathrm{Alt}^p(\mathcal{V}(M), \mathcal{E}(M)), \quad \text{cf. } (1.2),$$

where

$$\mathrm{Alt}^p(\mathcal{V}(M), \mathcal{E}(M)) = \{ \alpha : \mathcal{V}(M)^p \to \mathcal{E}(M) \mid \alpha \ \mathcal{E}(M)\text{-linear and alternating} \}.$$

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* Übung: Show (1.3) for $M \subset \mathbb{R}^n$ open, i.e. locally.

The differential $d: \mathcal{E}(M) \to \Omega^1(M)$ is extended to the EXTERIOR DERIVATIVE

$$d : \Omega^p(M) \to \Omega^{p+1}(M)$$

by defining $d\omega \in \mathrm{Alt}^{p+1}(\mathfrak{X}(M), \mathcal{E}(M))$ for $\omega \in \Omega^p(M)$ as

$$d\omega(X_0, \dots, X_p) := \sum_{j=0}^{p} (-1)^j L_{X_j}\left(\omega(X_0, \dots, \hat{X}_j, \dots X_p)\right)$$

[D] $$+ \sum_{1 \le j < k \le p} (-1)^{j+k} \omega\left([X_j, X_k], X_0, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots X_p\right)$$

for $X_0, X_1, \dots X_p \in \mathfrak{X}(M)$.

In local coordinates: For $\omega \in \Omega^p(M)$ with

$$\omega|_U = \omega_{i_1 \dots i_p} dq^{i_1} \wedge \dots \wedge dq^{i_p} \quad , \omega_{i_1 \dots i_p} \in \mathcal{E}(U),$$

(Einstein summation!) one has

[L] $$d\omega|_U = d\omega_{i_1 \dots i_p} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_p}.$$

with

$$d\omega_{i_1 \dots i_p} = \frac{\partial \omega_{i_1 \dots i_p}}{\partial q^j} dq^j \qquad \text{(as above).}$$

In fact, according to the definition of $d\omega$

$$(d\omega)_{i_0 \dots i_p} = d\omega\left(\frac{\partial}{\partial q^{i_0}}, \dots \frac{\partial}{\partial q^{i_p}}\right)$$

$$= \sum_{j=0}^{p} (-1)^j \frac{\partial}{\partial q^{i_j}} \omega\left(\frac{\partial}{\partial q^{i_0}}, \dots, \widehat{\frac{\partial}{\partial q^{i_j}}}, \dots \frac{\partial}{\partial q^{i_p}}\right)$$

$$= \sum_{j=1}^{p} (-1)^j \frac{\partial}{\partial q^j} \omega_{i_0 \dots \hat{i_j} \dots i_p}$$

which agrees with inserting $\left(\frac{\partial}{\partial q^{i_0}}, \dots \frac{\partial}{\partial q^{i_p}}\right)$ into

$$d\omega_{i_1 \dots i_p} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_p}.$$

Therefore, the formula [D] is a globally valid formula which locally has the effect described in [L]. Altogether, [D] defines an $\mathbb{R}$-linear map

$$d : \Omega^p(M) \to \Omega^{p+1}(M)$$

with $df(X) = L_X f$ for 0-forms $f \in \Omega^0(M) = \mathcal{E}(M)$ and $X \in \mathcal{V}(M)$ that the two properties of the following proposition are satisfied (one can show that $\mathbb{R}$-linearity, $df(X) = Xf$ and the two properties below uniquely determines d):

(1.4) PROPOSITION:[*]

$$d \circ d = 0,$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta \quad (\omega \in \Omega^p, \eta \in \Omega^k).$$

Consequently, with $\Omega^0(M) := \mathcal{E}(M)$,

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \quad \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

is a complex.

$\omega \in \Omega^k(M)$ is defined to be CLOSED, if $d\omega = 0$.
(closed = "geschlossen")
Each $\omega \in \Omega^n(M)$, $n = \dim M$, is closed. This follows from $\Lambda^p T_a M = \{0\}$ für $p > n$.

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* Übung: Check!

$\omega \in \Omega^k(M)$ is defined to be EXACT ("exakt") if $\omega = d\alpha$ for a suitable $\alpha \in \Omega^{k-1}(M)$.

In a convex open subset $M \subset \mathbb{R}^n$, every closed k-form is exact (Lemma of POINCARÉ). In general, one defines the de Rham groups

$$H^k_{dR}(M,\mathbb{R}) := \{\omega \in \Omega^k(M) \mid d\omega = 0\} / \{d\alpha : \alpha \in \Omega^{k-1}(M)\}.$$

Here, $\Omega^{-1}(M) = \{0\}$. Note, that $H^1_{dR}(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) \cong \mathbb{R}$.

We now come to the fundamental notion of classical mechanics:

(1.5) DEFINITION: A SYMPLECTIC MANIFOLD is a manifold M together with a SYMPLECTIC FORM ω, $\omega \in \Omega^2(M)$, that is

1° $\forall a \in M: \omega(a): T_aM \times T_aM \to \mathbb{R}$ is non-degenerate,

2° $d\omega = 0$.

From 1° we deduce for a symplectic manifold (M,ω)

$$\dim M = \dim_{\mathbb{R}} T_aM \in 2\mathbb{N} \text{ is even,}$$

as in our fundamental example $M = P = T^*\Omega$, $\Omega \subset \mathbb{R}^n$, open. Moreover, $\operatorname{rk} \omega(a) = \dim M$, for the rank "rk". Since the symplectic form ω is non-degenerate it induces a diffeomorphism

$$\omega^b : TM \to T^*M$$

by

$$\omega^b(X)(Y) := \omega(X,Y), \quad X,Y \in \mathcal{V}(M).$$

$\omega^\flat$ commutes with the projection (i.e. $\tau = \tau^* \circ \omega^\flat$) and is $\mathbb{R}$-linear on the fibres. In particular, for each $a \in M$,

$$T_a M \ni v \longmapsto \omega^\flat(v) \in T_a^* M$$

is an isomorphism of $\mathbb{R}$-vector spaces, in fact $\omega^\flat$ is an isomorphism of vector bundles over id_M.

In local coordinates: Let $\omega|_U = \omega_{ij}\, dq^i \wedge dq^j$. For $X = X^j \frac{\partial}{\partial q^j}$ one obtains

$$\omega^\flat(X) = (\omega_{ij} X^i)\, dq^j =: \omega^\flat(X)_j\, dq^j =: X_j\, dq^j .$$

Hence, $\omega^\flat$ is "lowering" the index of X^j.

The inverse of $\omega^\flat$,

$$\omega^\# := (\omega^\flat)^{-1} : T^*M \to TM$$

is, therefore, "raising" the index: If $\alpha|_U = \alpha_i\, dq^i$, $\alpha \in \Omega^1(M)$ we obtain

$$\omega^\#(\alpha)|_U = \omega^{ij} \alpha_i \frac{\partial}{\partial q^j} =: \omega^\flat(\alpha)^j \frac{\partial}{\partial q^j} =: \alpha^j \frac{\partial}{\partial q^j},$$

where (ω^{ij}) is the inverse matrix to (ω_{lk}), $\omega|_U = \omega_{lk}\, dq^l \wedge dq^k$.

(1.6) <u>DEFINITION:</u> Let (M, ω) be a symplectic manifold. To every $H \in \mathcal{E}(M)$ there corresponds the HAMILTONIAN VECTOR FIELD

$$X_H := \omega^\# \circ dH .$$

The diagram

$$\begin{tikzcd} T^*M \arrow[rr, "\omega^\#"] & & TM \\ & M \arrow[ul, "dH"] \arrow[ur, dashed, "X_H"'] & \end{tikzcd}$$

is commutative. Hence, $dH = \omega^\flat X_H$, i.e.:
X_H is the uniquely defined vector field satisfying

$$\omega(X_H, Y) = dH(Y) = L_Y H \qquad \text{for all } Y \in \mathcal{V}(M).$$

In local coordinates

$$X_H|_U = \omega^{ij} \frac{\partial H}{\partial q^i} \frac{\partial}{\partial q^j} = \frac{\partial H}{\partial q^i} \omega^{ij} \frac{\partial}{\partial q^j}$$

(X, ω, H) is called a HAMILTONIAN SYSTEM.

A MOTION ("Bewegung") of the system (M, ω, H) is a curve $x \in \mathcal{E}(I, M)$, $I \subset \mathbb{R}$ an open interval, such that

$$\dot{x} = X_H(x).$$

In local "canonical coordinates" (see below, DARBOUX's theorem), i.e. $\omega|_U = dq^j \wedge dp_j$, $\dot{x} = X_H(x)$ is equivalent to

$$\dot{q}(t) = \frac{\partial H}{\partial p}, \quad \dot{p}(t) = -\frac{\partial H}{\partial q},$$

if $q(t) := q(x(t))$, $p(t) = p(x(t))$, where $(q, p) : U \to V \subset \mathbb{R}^n \times \mathbb{R}^n$ is the canonical chart.

(1.7) EXAMPLE: Cotangent bundle.
Let Q be a manifold of dimension n. $M := T^*Q$ is $2n$-dimensional. A chart $\varphi = (q^1, \dots q^n) : U \to V \subset \mathbb{R}^n$ of Q induces a bundle chart on T^*U:

$$\tilde{\varphi} = (q^1, \dots q^n, p_1, \dots p_n) : T^*U \longrightarrow V \times \mathbb{R}^n,$$

$$p_k(\alpha) = \alpha_k \qquad \text{for } \alpha = \alpha_k dq^k \in T_a^*M.$$

Hence, $p_k(\alpha) = \alpha(\frac{\partial}{\partial q^k}) \in \mathbb{R}^n$. (Note, that the collection of all these bundle charts defines the $\mathcal{C}^\infty$-structure on T^*M.)

The LIOUVILLE form on T^*U is by definition

$$\lambda := p_j dq^j$$

More generally, for $\alpha \in T_a^*Q \subset M$ and $X \in T_\alpha M$ we can define the Liouville form globally by

$$\lambda(\alpha)(X) := \alpha(T_\alpha\tau^*(X)) = \langle \alpha, T_\alpha\tau^*(X)\rangle,$$

where $\tau^* : T^*Q \to Q$ is the canonical projection and $T_\alpha\tau^* : T_\alpha M \to T_a Q$ the derivative of τ^* at α (see page 8).

Then $\lambda \in \Omega^1(M)$. With respect to a bundle chart $\tilde{\varphi} : T^*U \to V \times \mathbb{R}^n$ one sees

$$\lambda|_{T^*U} = p_j dq^j$$

The corresponding symplectic form is

$$\omega := -d\lambda \in \Omega^2(M)$$

with

$$\omega|_{T^*U} = dq^j \wedge dp_j .$$

The local expression indeed shows that ω is non-degenerate and closed.

Remark: In the case of a cotangent bundle $M = T^*Q$ the form ω is also exact. The form $-\lambda$ is called a SYMPLECTIC POTENTIAL.

In general, since a symplectic form ω is closed by definition, ω has potentials locally, i.e. for each point $a \in M$ there is a neighbourhood U (e.g. a coordinate neighborhood U with chart $\varphi : U \to V \subset \mathbb{R}^{2n}$, V convex) such that there exists $\alpha \in \Omega^1(U)$ with $d\alpha = \omega$. A global symplectic potential always exists if $H^2_{dR}(M, \mathbb{R}) = 0$.

Locally all symplectic manifold look like open subspaces of (T^*Q, ω) for $Q \subset \mathbb{R}^n$ open:

(1.8) THEOREM: (DARBOUX's THEOREM) Every point $a \in M$ of a symplectic manifold (M, ω) has a neighbourhood $U \subset M$ and a chart

$$\varphi = (q^1 .. q^n ; p_1, \dots p_n) : U \longrightarrow V \subset \mathbb{R}^n \oplus \mathbb{R}^n ,$$

such that $\omega|_U = dq^j \wedge dp_j$. The q^j, p_k are called CANONICAL COORDINATES.

As we mentioned above, in canonical coordinates the equations of motion of a Hamiltonian system (M, ω, H) have the form

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

<u>POISSON BRACKET</u>

As before in the flat case of $P = T^*\Omega$, $\Omega \subset \mathbb{R}^n$ open, the symplectic form ω defines a Poisson bracket (and vice versa) for a symplectic manifold by

$$\{F, G\} := \omega(X_F, X_G) \quad , \quad F, G \in \mathcal{E}(M).$$

The equations of motion with respect to a Hamiltonian system (M, ω, H) can again be written in Poisson form

$$\dot{F} = \{F, H\}, \quad \text{cf. } (1.1).$$

As a result, every F with $\{F, H\} = 0$ is a first integral!

Besides the general formula

$$\{F, G\} = i_{X_F}\omega(X_G) = dF(X_G) = L_{X_G} F$$

we have the description in local coordinates $q: U \to V$, $q = (q^1, \dots \quad q^{2n})$:

$$\{F,G\}|_U = \frac{\partial F}{\partial q^i}\,\omega^{ij}\,\frac{\partial G}{\partial q^j}.$$

This follows from

$$\{F,G\}|_U = \omega\left(\frac{\partial F}{\partial q^i}\,\omega^{ij}\frac{\partial}{\partial q^j},\ \frac{\partial G}{\partial q^k}\,\omega^{kl}\frac{\partial}{\partial q^l}\right) = \frac{\partial F}{\partial q^i}\,\omega^{ij}\omega_{jk}\omega^{kl}\,\frac{\partial G}{\partial q^k}.$$

We conclude section 1 with the following basic results on the Poisson bracket:

(1.9) PROPOSITION: The Poisson bracket $\{\ ,\ \}$ of a symplectic manifold (M,ω) is a Lie bracket i.e. $\mathcal{E}(M)$ with $\{\ ,\ \}$ is a Lie algebra over $\mathbb{R}$:

1° $\{\ ,\ \} : \mathcal{E}(M)\times\mathcal{E}(M) \longrightarrow \mathcal{E}(M)$ is bilinear over $\mathbb{R}$.

2° $\{F,G\} = -\{G,F\}$ ($\{\ ,\ \}$ is alternating)

3° $\{F,\{G,I\}\} + \{G,\{I,F\}\} + \{I,\{F,G\}\} = 0$

for all $F,G,I \in \mathcal{E}(M)$. ($\{\ ,\ \}$ fulfills the JACOBI-identity)

In addition:

4° $\{F,GI\} = G\{F,I\} + \{F,G\}I$
$= G\{F,I\} + I\{F,G\}$ (product rule, $\{F,\ \} : \mathcal{E}(M) \to \mathcal{E}(M)$ is a derivation on the $\mathbb{R}$-algebra $\mathcal{E}(M)$.)

5° G is constant, if $\{F,G\} = 0$ for all $F \in \mathcal{E}(M)$. (completeness; only correct for connected manifolds. In general: $dG = 0$, i.e. G is locally constant.)

The proof of 3° is essentially equivalent to the next proposition, but can also be shown by using the local expression for the Poisson bracket.

(1.10) <u>PROPOSITION</u>: The formation $F \overset{\Phi}{\longmapsto} -X_F$, $F \in \mathcal{E}(M)$, is a Lie algebra homomorphism

$$\Phi : \mathcal{E}(M) \longrightarrow \mathcal{V}(M),$$

i.e. $[X_F, X_G] = -X_{\{F,G\}}$ for all $F, G \in \mathcal{E}(M)$.

□ <u>Proof</u> of (1.10): For all $F, G, H \in \mathcal{E}(M)$ & $X = X_F$, $Y = X_G$:

$$L_{[X,Y]} H = L_{X_F} L_{X_G} H - L_{X_G} L_{X_F} H .$$

Applying $L_{X_F} I = \{I, F\}$ (recall that

$$L_{X_F} I = dI(X_F) = \omega(X_I, X_F) = \{I, F\})$$

we obtain

$$\begin{aligned} L_{[X,Y]} H &= \{L_{X_G} H, F\} - \{L_{X_F} H, G\} \\ &= \{\{H,G\},F\} - \{\{H,F\},G\} \\ &= \{F,\{G,H\}\} + \{G,\{H,F\}\} \qquad (\text{Jacobi}) \\ &= -\{H,\{F,G\}\} = -L_{X_{\{F,G\}}} H \end{aligned}$$

Hence, $[X_F, X_G] = -X_{\{F,G\}}$. □

As a result the hamiltonian vector fields

$$\mathfrak{h}(M) := \{ X \in \mathcal{V}(M) \mid \exists H \in \mathcal{E}(M) : X = X_H \}$$

form a Lie subalgebra of the Lie algebra $\mathcal{V}(M)$ of vector fields. The kernel $\operatorname{Ker} \Phi \subset \mathcal{E}(M)$ of $\Phi : \mathcal{E}(M) \to \mathcal{V}(M)$ consists of the locally constant functions on M. Hence, for connected manifolds M we have $\mathbb{R} = \operatorname{Ker} \Phi$ and we obtain the following exact sequence of Lie algebras

$$0 \to \mathbb{R} \to \mathcal{E}(M) \xrightarrow{\Phi} \mathfrak{h}(M) \to 0 .$$

A vector field X is called LOCALLY HAMILTONIAN if for all $a \in M$ there exists a neighbourhood U and $H \in \mathcal{E}(U)$ with $X|_U = X_U$. Let $\mathfrak{lh}(M)$ denote the set of all locally hamiltonian vector fields. $\mathfrak{lh}(M) \subset \mathcal{V}(M)$ is, of course, a linear subspace.

(1.11) PROPOSITION: Let (M, ω) be a symplectic manifold.

1° $X \in \mathfrak{lh}(M) \iff d(i_X \omega) = 0 \iff L_X \omega = 0$.

2° $X, Y \in \mathfrak{lh}(M) \Rightarrow [X, Y] = - X_{\omega(X,Y)}$

3° $\mathfrak{lh}(M)$ is a Lie subalgebra of $\mathfrak{h}(M)$ and $\mathfrak{h}(M)$ is an ideal in $\mathfrak{lh}(M)$.

4° $\mathfrak{lh}(M) = \mathfrak{h}(M)$ whenever M is simply connected.

☐ *Proof.* 1° For $\eta \in \Omega^k(M)$ and $X \in \mathcal{V}(M)$ the $(k-1)$-form $i_X\eta$ is defined by

$$i_X\eta(X_1, \dots X_{k-1}) := \eta(X, X_1, \dots X_k),$$

and the LIE DERIVATIVE $L_X : \Omega^k(M) \to \Omega^k(M)$ is given by $L_X := i_X d + d i_X$. To show 1° we first observe $L_X\omega = d i_X \omega$. For any $a \in M$ we find a chart $q : U \to V$ such that $a \in U$ and V is an open and convex subset of $\mathbb{R}^{2n}$. Now $i_X\omega_{|U} = dH$ for a suitable $H \in \mathcal{E}(U)$ implies $d(i_X\omega) = 0$, and hence $L_X\omega = 0$. Conversely, $d(i_X\omega) = 0$ implies (Lemma of Poincaré) that $i_X\omega$ is of the form $i_X\omega = dH$, hence $X|_U = X_H$.

2° The assertion has only to be shown locally, hence we can assume $X = X_F$, $Y = X_G$. The result has been shown in the above proof for (1.10).

3° follows directly from 2° and 4° follows from the fact that $H^1_{dR}(M, \mathbb{R}) = 0$ for simply connected M.